TECH 350: DSP

Class VIII: Transfer Functions, Complex Numbers, Prerequisites for Fourier Transform

Transfer Functions

Give x(n) as input and y(n) as output, the ***transfer function,*** H(z), may be written as:

$$H(z) = \frac{Y(z)}{X(z)}$$

Where X(z) = Z(x(n)) and Y(z) = Z(y(n)) (z-Transforms of x(n) and y(n))

This follows from the fact that the impulse response convolved with the input is the output:

H(z) is equal to the z-Transform of the impulse response h(n) (neat!)

Transfer Functions

Transfer Function*: provides an algebraic representation of an LTI filter in the *frequency domain

Before we look at an example:

There are two different conventions for denoting delays and advances:

Laplace: uses positive powers (e.g. Z^2) for delay and negative powers (e.g. Z^{-2})

Engineering: uses negative powers (e.g. Z^{-2}) for delay and positive powers (e.g. Z^2)

I felt that positive powers was useful for explaining the z-Transform, but we will proceed with the engineering convention moving forward

Transfer Function: Example 1

Difference Equation: $y(n) = a\,x(n) + b\,x(n-1) + c\,y(n-1)$ (remember a, b, and c are coefficients)

Transfer Function:

first, get the z-Transform of the part of the diff. equation with x(n) (for the denominator):

$Z(\,a\,x(n) + b\,x(n-1)\,) = 1a + bZ^{-1} = a + bZ^{-1}$

second, get the z-Transform of the part of the diff. equation with y(n) (for the numerator):

$Z(\,c\,y(n-1)\,) = 1 - cZ^{-1}$ (y(n) is on the left side, we move c y(n-1) to the right)

third, combine these:

$H(n) = 1 - cZ^{-1} / a + bZ^{-1}$

Transfer Function: Example 2

Difference Equation (biquad): $y(n) = a_0x(n) + a_1x(n-1) + a_2x(n-2) - b_1y(n-1) - b_2y(n-2)$

first, get the z-Transform of the part of the diff. equation with x(n) (for the denominator)

second, get the z-Transform of the part of the diff. equation with y(n) (for the numerator)

third, combine these

Zeros + Poles (after JOS)

We may write a general form of the transfer function (with the leading coefficient in the numerator called g) as such:

$$H(z) = g\frac{1 + \beta_1 z^{-1} + \cdots + \beta_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

We may then factor the numerator and denominator to obtain:

$$H(z) = g\frac{(1 - q_1 z^{-1})(1 - q_2 z^{-1})\cdots(1 - q_M z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})\cdots(1 - p_N z^{-1})}.$$

If z is set to any of the numbers q_1, q_2, …q_M (what are called the *zeros*), the transfer function evaluates to 0.

As z approaches any of p_1, p_2, …p_M (what are called the *poles*), the transfer function approaches infinity.

Zeros + Poles (after JOS)

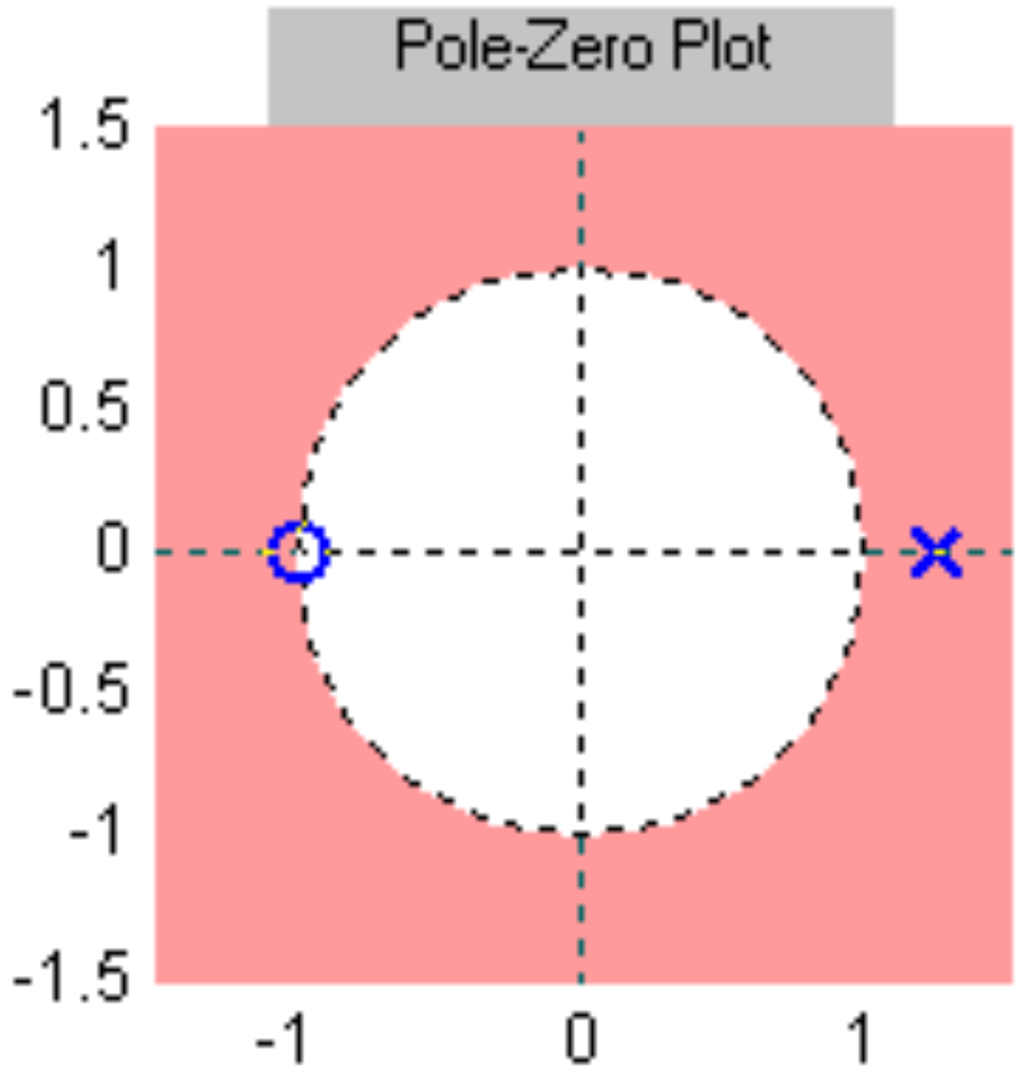
For Computer Musicians:

That's why, if you've screwed around with filters and coefficients, they've either suddenly gone silent (and need some coddling to get sounding again) (**zeros**) or they explode into deafening feedback (**poles**)

We can additionally use zeroes and poles to describe filters, e.g.:

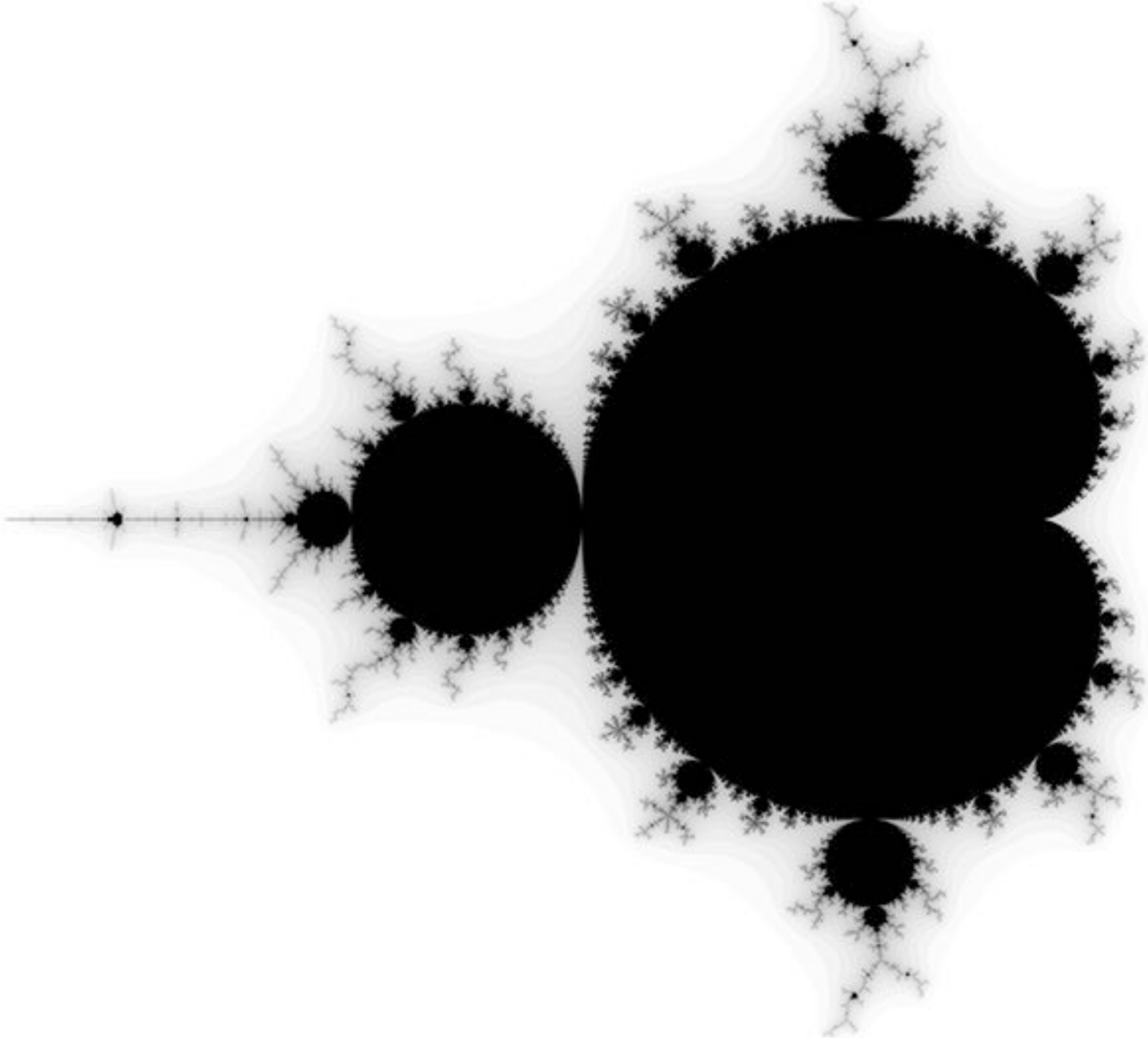
"The biquad filter is a two-pole, two-zero filter"

$$H(z) = g\frac{1+\beta_1 z^{-1}+\beta_2 z^{-2}}{1+a_1 z^{-1}+a_2 z^{-2}}$$



Plot in the z-Domain that shows where the filter goes to zero (zeros, Os) and trends to infinity (poles, Xs)

The Mandelbrot Set



$$z_{n+1} = z_n^2 + z_0$$

What is *z*, though?

z is a ***complex number***, a topic that I've been avoiding, but that we necessarily need to dig into a bit before we move on to an actual definition of the Fourier Transform.

Since this class is targeted at electronic musicians, I'm going to introduce complex numbers in bite-size pieces, although some of you already eat them for breakfast.

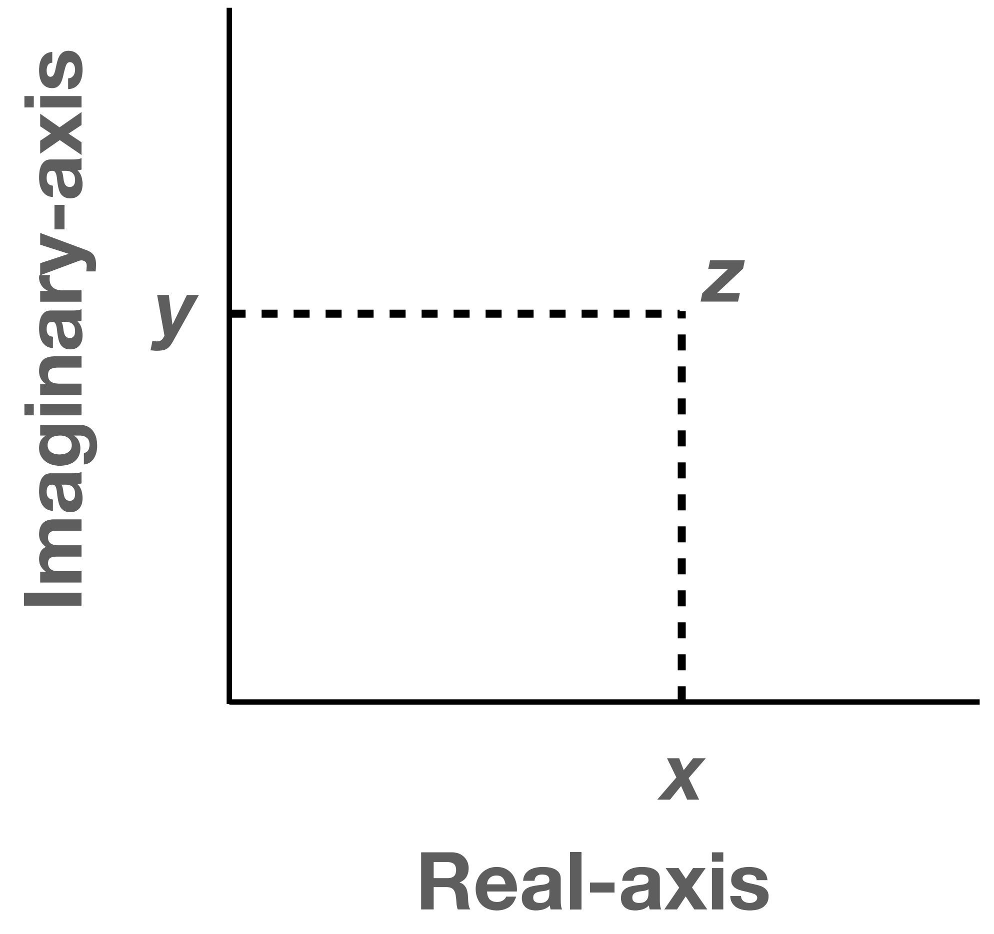
A complex number (in its ***rectangular form***) is written as:

z = x + yj,

where x is the ***real*** part of the number and y is the ***imaginary*** part

j, which is equivalent to √-1, is called the ***unit imaginary number***

(NOTE: i is used outside of an engineering context)



A Taste of Complex Numbers

Converting the rectangular form ($z = x + yj$) to the polar, we get the following:

$z = r\cos(\theta) + j\,r\sin(\theta)$,

where θ is an angle and $r = |z| = \sqrt{x^2 + y^2}$

(i.e., r is the distance from the origin of the complex number)

This is made more useful by taking advantage of Euler's identity,

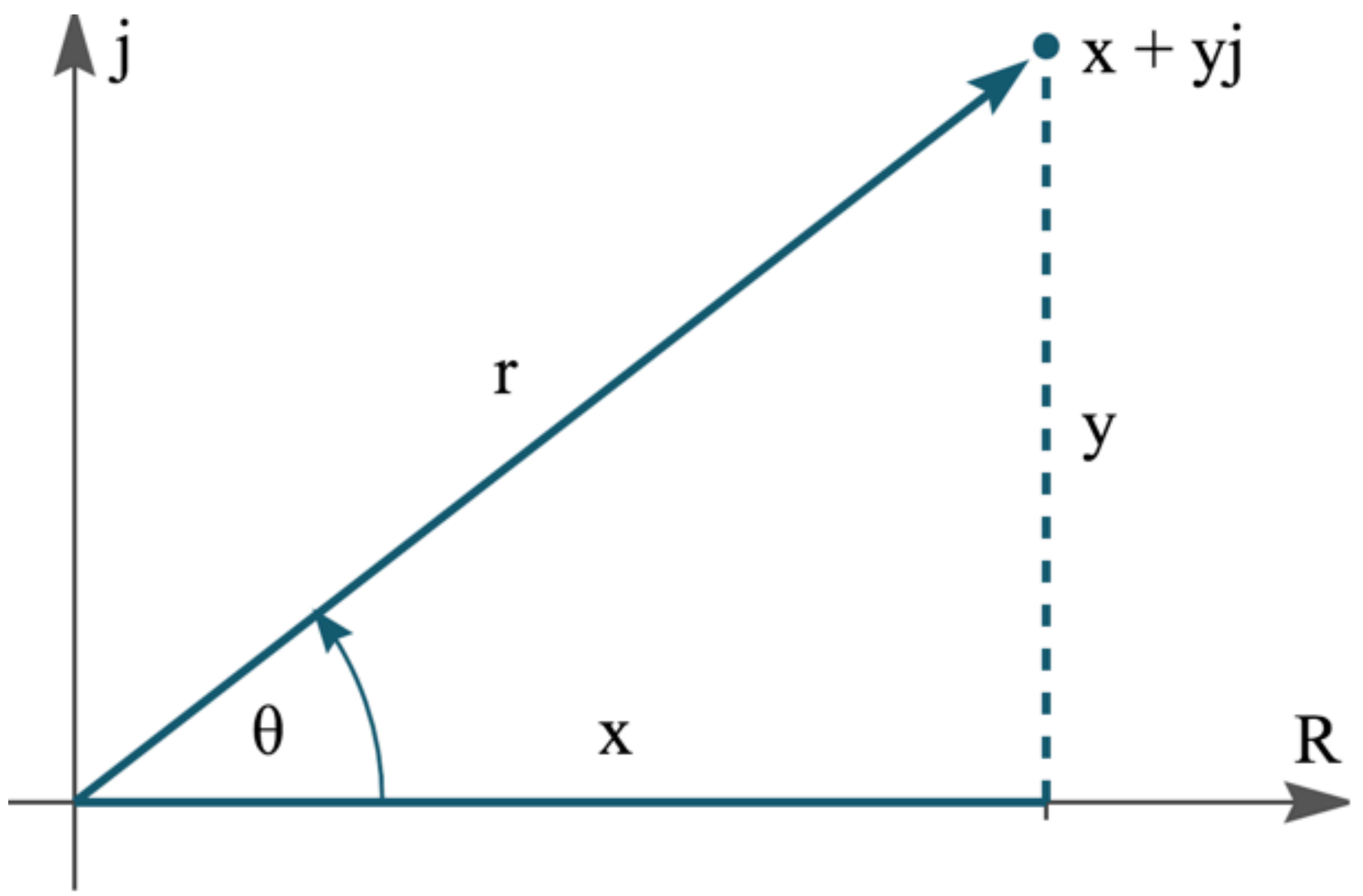
$e^{j\theta} = \cos(\theta) + j\sin(\theta)$,

where e is Euler's number, the base of the natural logarithm,

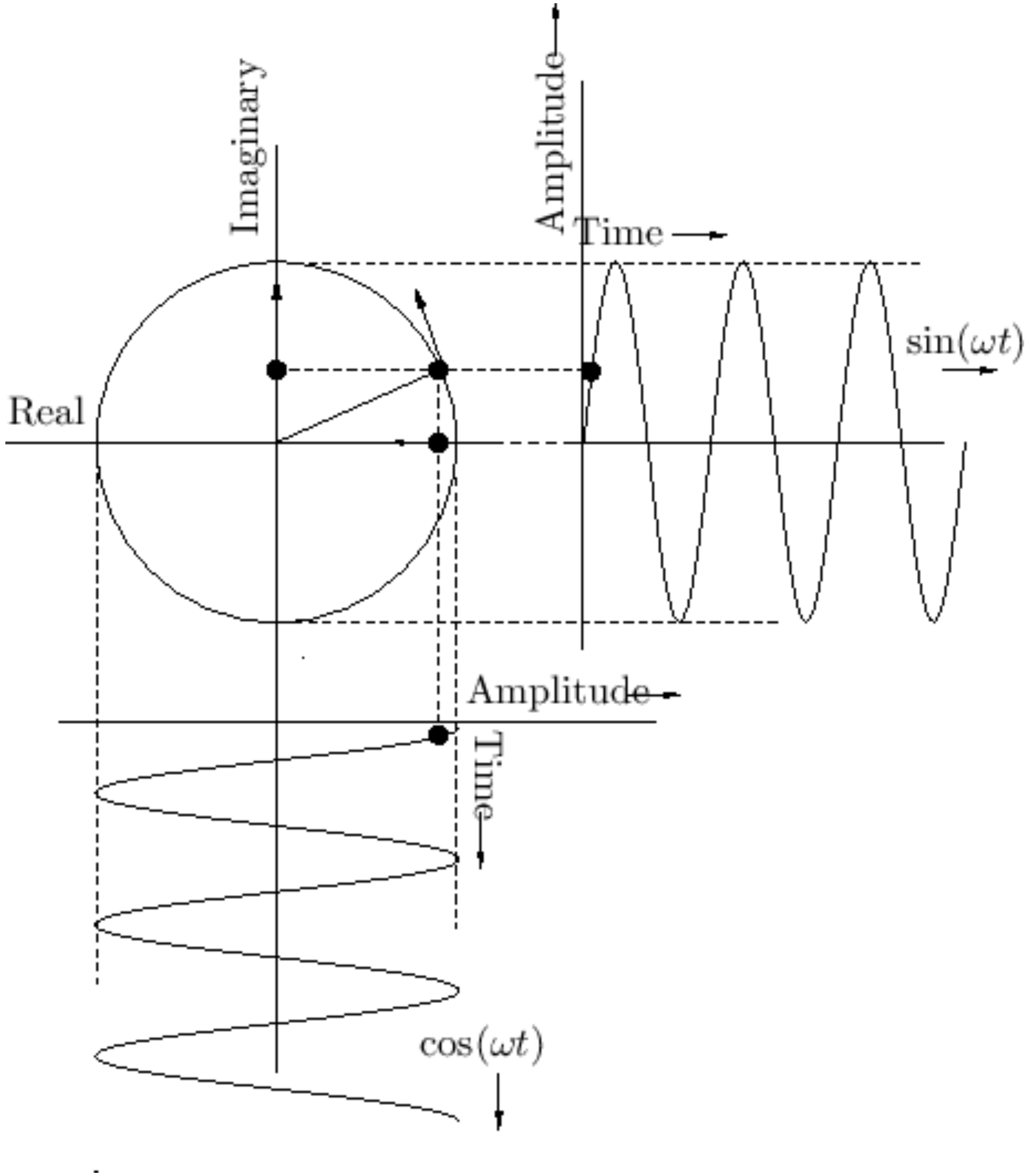
to create the form we'll use to explore the Fourier Transform:

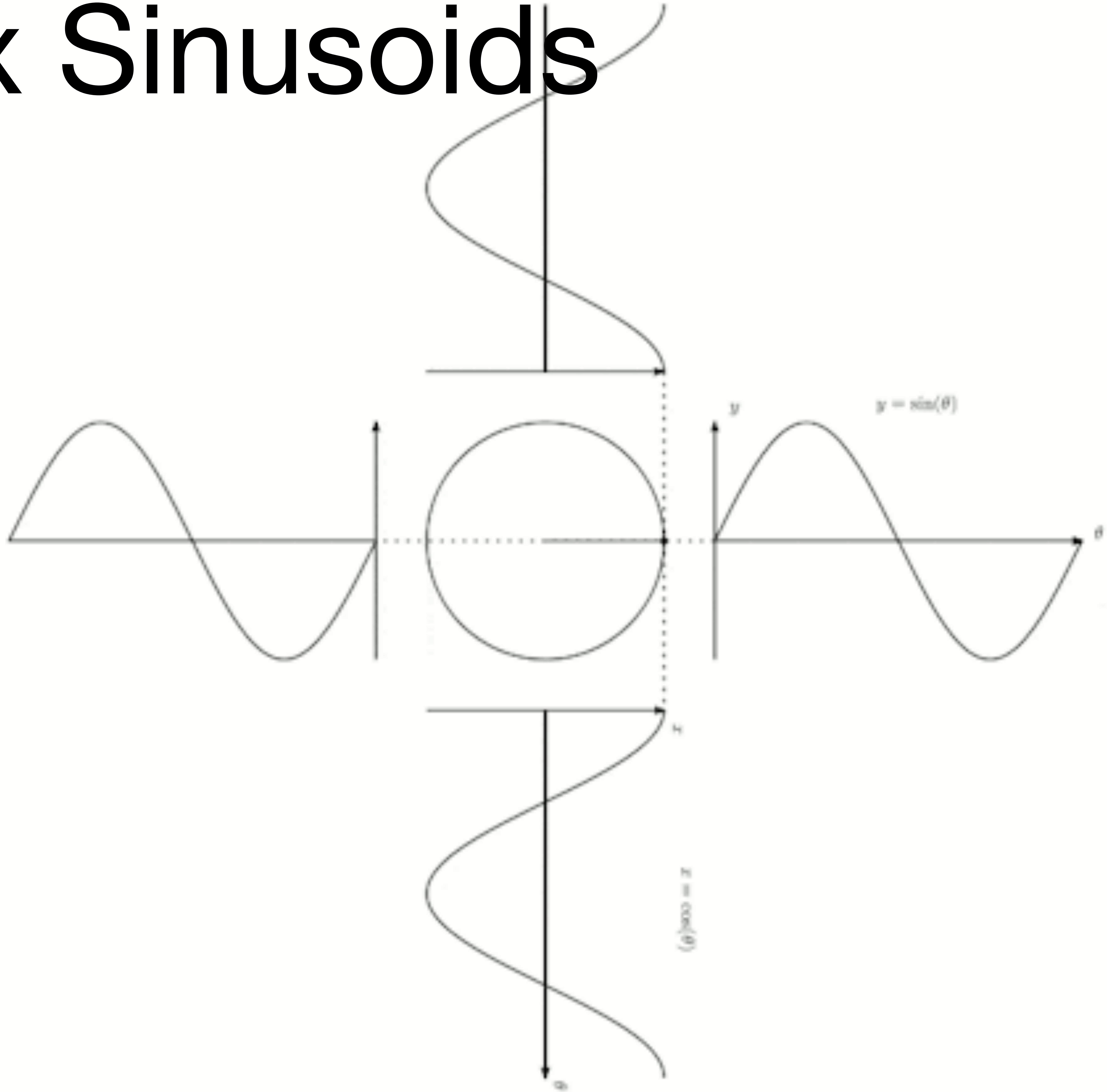
$z = re^{j\theta}$,

the ***polar form (AKA exponential form)*** of the complex number.



Complex Sinusoids

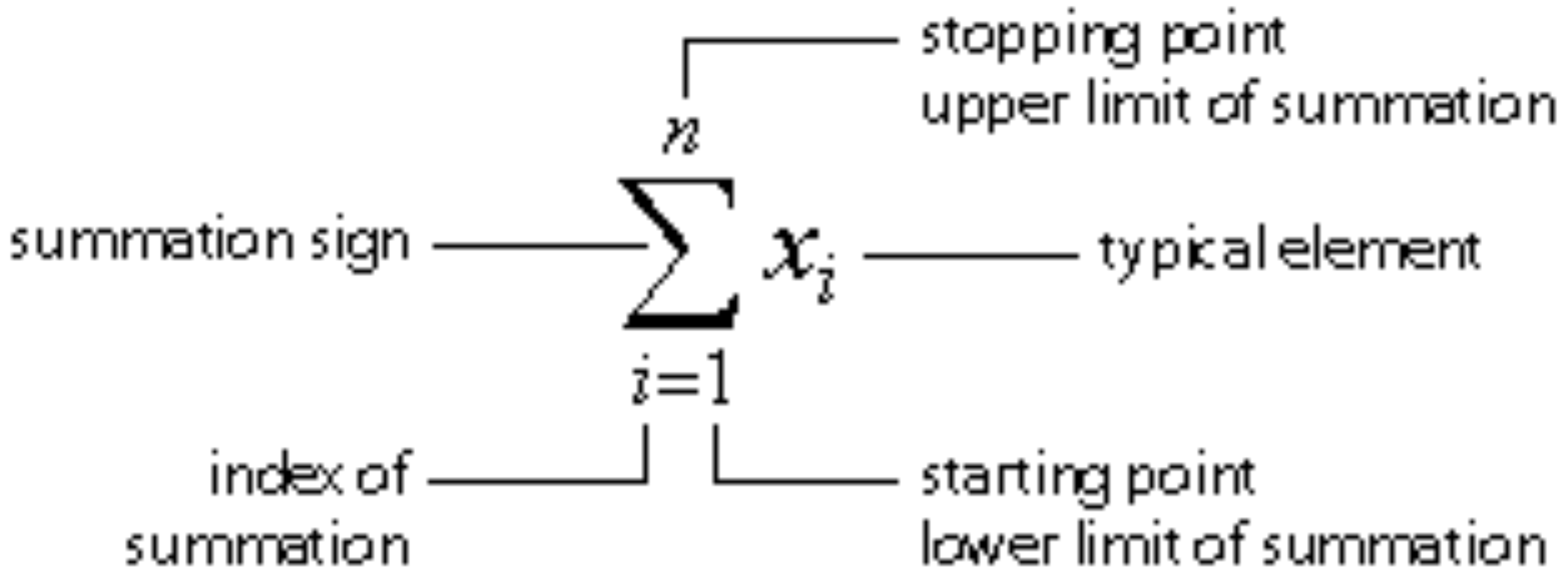




Mathematical Symbology: Summation

Summing: a lot of the procedures we've been doing (convolution, calculating the output of a filter) involve summing terms.

As shorthand for summation (adding together) of terms, we use the following (Big Sigma):



For example,

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

The Fourier Transform: Take 0

Jean Baptiste Joseph Fourier

Late 1700s researcher on periodic waves and their analysis

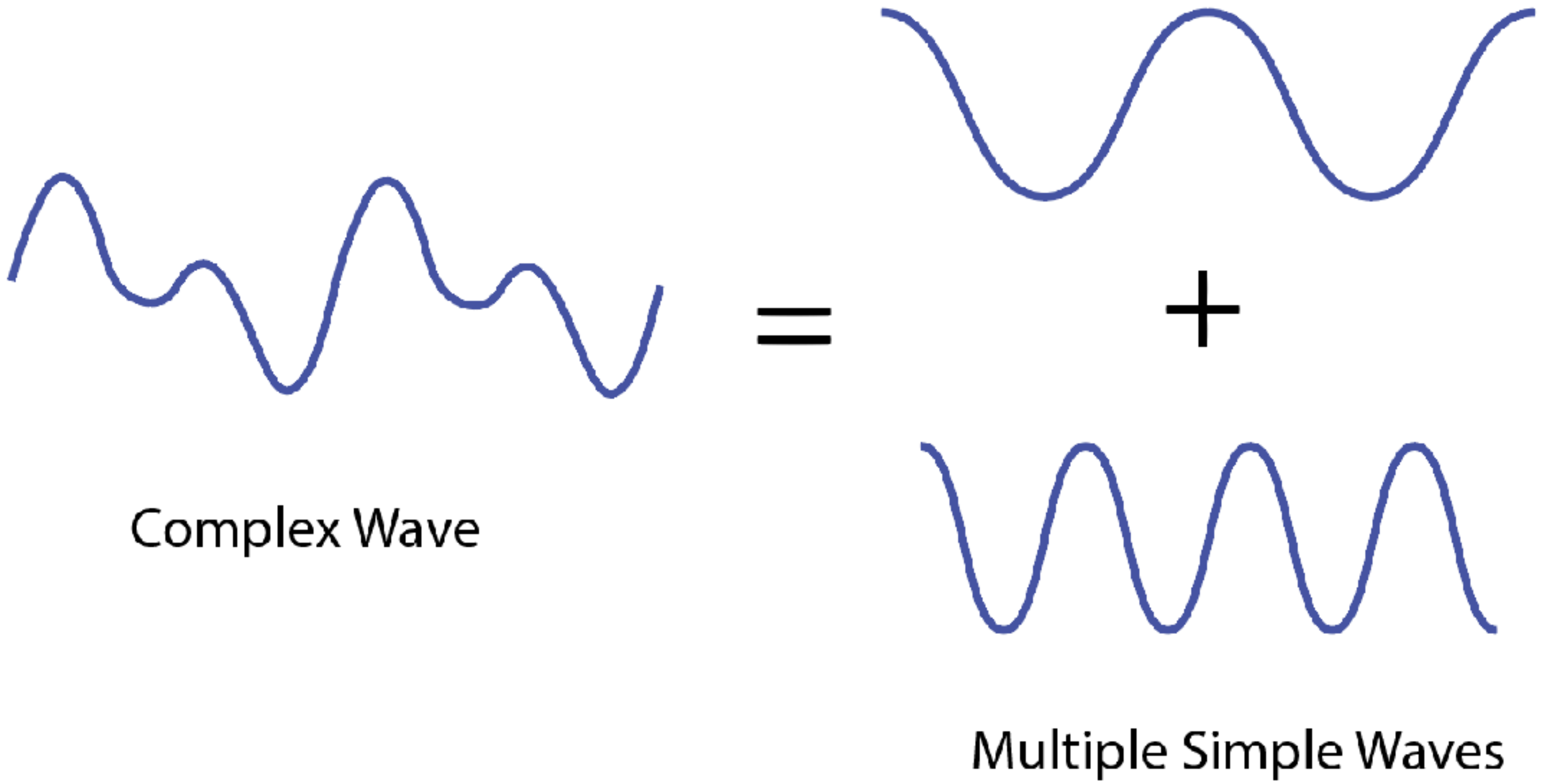
Studied the conductive diffusion of heat

Scientific advisor to Napoleon

Fourier's Theorem:

A complex periodic sound can be decomposed into a set of simple waves

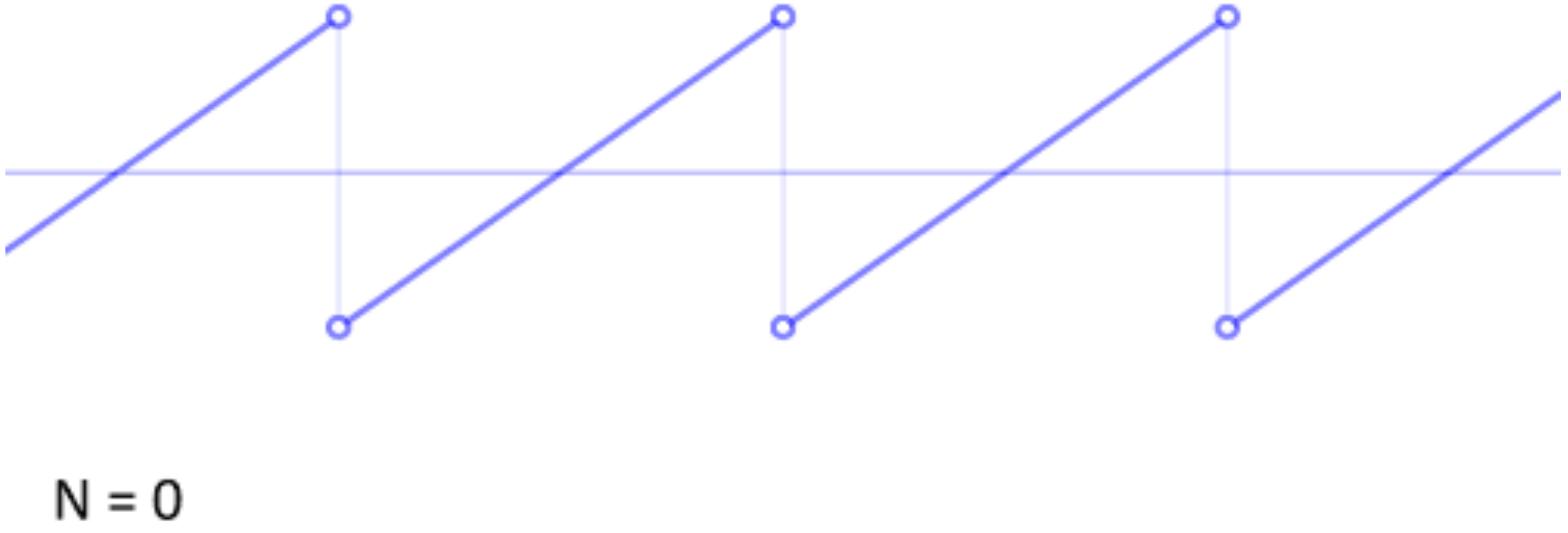
Deconstructing a complex wave (Fourier analysis)



Fourier's Theorem:

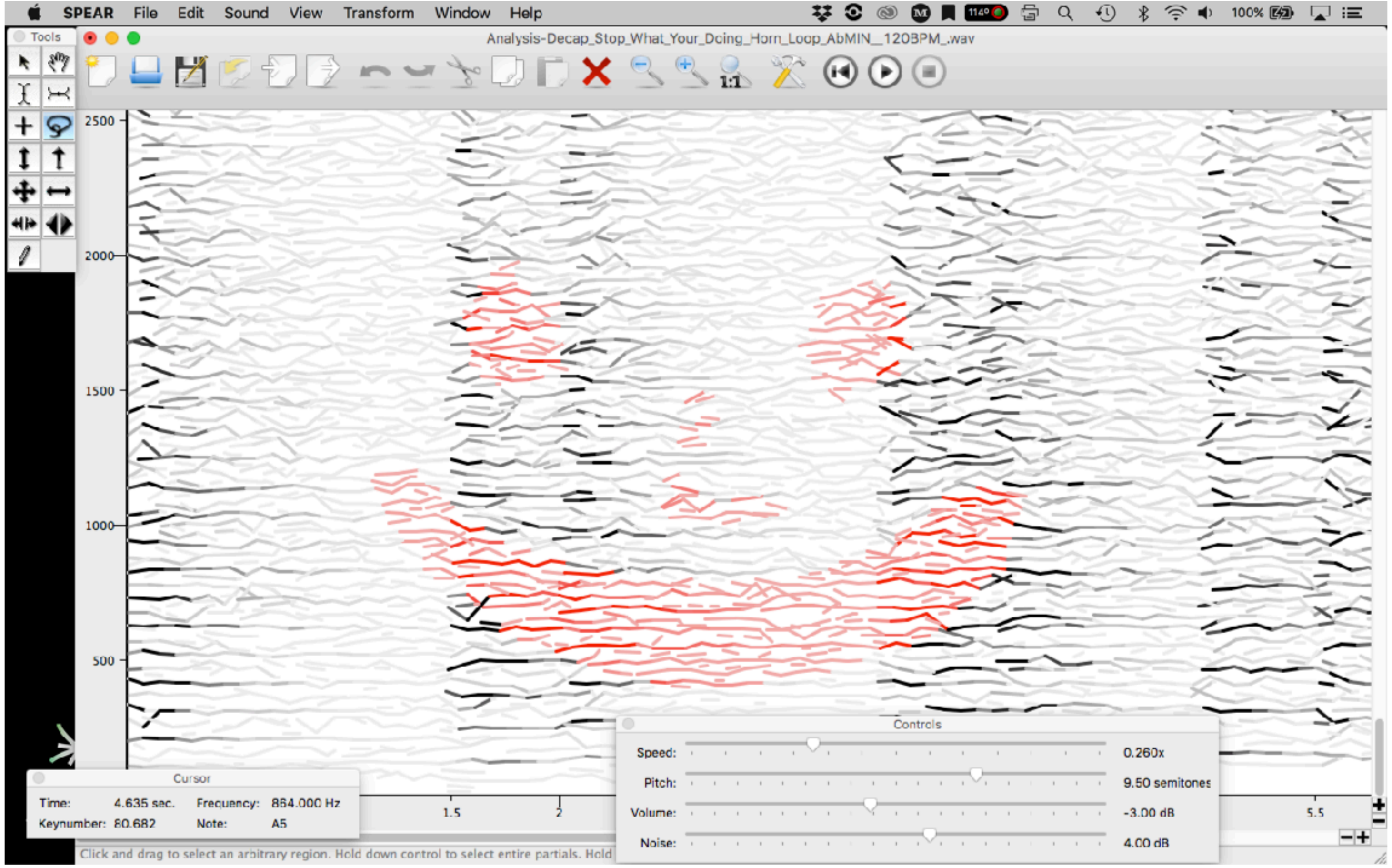
A complex periodic sound can be decomposed into a set of simple waves

Summing simple waves (additive synthesis)



Approximating a sawtooth wave using multiple simple (sine) waves

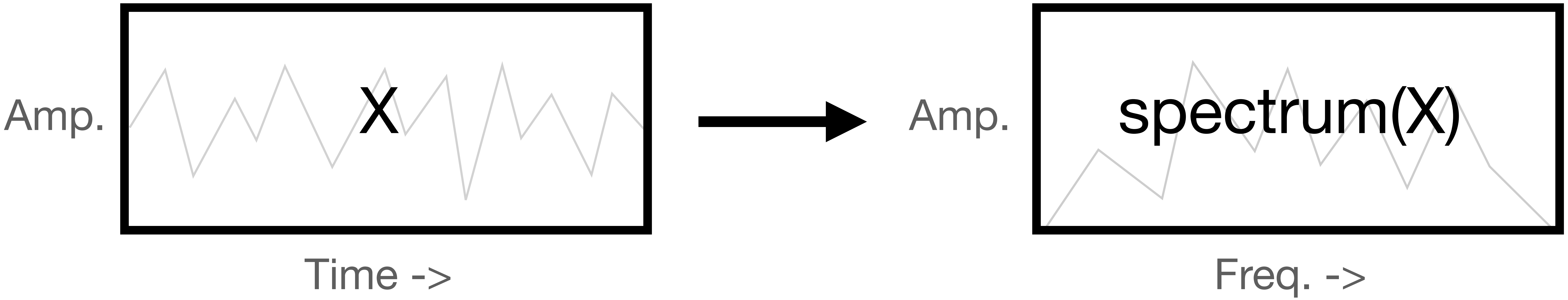
SPEAR (SPectral Editing, Analysis, and Resynthesis)



http://www.klingbeil.com/spear/

The Fourier Transform: Take I

Goal: *Given some signal X, we want to decompose X into its constituent frequencies, that is, we want the spectrum of X.*



The result (the Fourier Transform of X) will give us two types of information:

1. How much a particular frequency sinusoid is present (its magnitude) and, additionally,
2. Where in the cycle of the sinusoid it begins (its phase offset)

Remember that we get these two different (but intimately related) types of information!

LTI Systems Review (Concepts we've Covered)

System Properties

Signal Flow Diagrams

Difference Equations

Sine-Wave Analysis

Frequency Response

Filter Parameters

Filter Topologies

Comb + Allpass Filters

Digital Algorithm Reverb Basics

Impulse Response (FIR vs. IIR)

z-Transform

The Convolution Theorem

Transfer Functions

Zeroes + Poles

Complex Numbers Basics